IIA in Separable Matching Markets

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Abstract

Independence of Irrelevant Alternatives has figured prominently in social choice and in decision-making under risk. Its extension to two-sided markets is not obvious. We redefine IIA in models of matching with transferable utility; we also define Independence of Irrelevant Type Splits (ITS) and Irrelevance of Type Aggregation (ITA). We discuss these properties in four models: (i) Choo and Siow 2006 (ii) a 2-nest logit model; (iii) a new class of Generalized Random Coefficients models; and (iv) the nonseparable model of Dagsvik 2000.

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Introduction

Independence of irrelevant alternatives has figured prominently in at least three different contexts in economics. Its first appearance dates back to Nash's 1950 paper on the bargaining problem. When discussing the representation of choice by a utility function, Nash (1950, property 7) required that an optimal choice cannot become non-optimal when the choice set is restricted to a subset that contains it. In formal terms, let $C(S)$ denote the set of optimal choices set from a set of alternatives. Nash required that if $C(S) \subset T \subset S$, then $C(T) = C(S)$. While this condition is necessary and sufficient for C to be represented by a binary relation, it may not be acyclic[1]. In the monograph that gave birth to social choice theory, Arrow (1951) defined independence of irrelevant alternatives as imposing that social preferences between a pair of alternatives only depend on the collection of individual preferences over that pair. The next step was taken by Luce and Raiffa (1957) and Luce (1959) with decision-making under risk. Luce called independence from irrelevant alternatives his Axiom 1 (p. 6), which requires that probabilistic choice satisfy:

$$\text{if } R \subset S \subset T, \ \text{ then } P_T(R) = P_S(R)P_T(S)$$

where $P_T(S)$ is the probability that choice from T belongs to S. Luce described it as "one reasonable possibility" to extend to probabilistic choice the notion that adding new alternatives should not change preferences between existing alternatives. As is now well-known, Luce's IIA has stark consequences for stochastic choice: it implies the existence of a positive function v such that for any finite set S and any alternative $x \in S$,

$$P_S(x) = \frac{v(x)}{\sum_{y \in S} v(y)}.$$

If we define $u(x) = \ln v(x)$, then choice probabilities take the familiar "multinomial logit" form

$$P_S(x) = \frac{\exp(u(x))}{\sum_{y \in S} \exp(v(y))}.$$

[1] Arrow (1959) showed that if the domain of the choice correspondence C contains all finite subsets, then IIA is necessary and sufficient for the existence of a transitive complete binary relation that represents C.

The aim of this note is to explore the extension of IIA to a class of models of two-sided choice. We will focus on one-to-one bipartite matching models with perfecty transferable utility. In addition, we will restrict our analysis to markets in which the joint surplus is "separable" in the sense of Chiappori et al. (2017) and Galichon-Salanié (2016). Separability is a data-driven concept developed with empiricists in mind, following the pioneering work of Choo and Siow (2006). It assumes that conditional on observed "types", the characteristics of the agents that are unobserved by the analyst do not interact in the production of joint surplus. We consider this as a first step towards a more general inquiry into IIA and related properties in matching models.

Section 1 defines our setup and notation. We discuss concepts of IIA for matching models in section 2; after settling on a definition, we also define two other properties which we call Independence of Irrelevant Type Splits (ITS) and Irrelevance of Type Aggregation (ITA); there are directly motivated by the well-known blue bus/red bus example in the tradition of Debreu (1960). We show that ITS holds in a class of Generalized Random Coefficients models. Section 3 shows that in the Choo and Siow (2006) model, IIA holds but neither ITS not ITA do—a direct analog to the blue bus/red bus example for matching models. Finally, we provide various remarks and extensions in section 4.

1 Separable Matching with Transferable Utility

In all of the following, we consider frictionless bipartite matching with perfectly transferable utility (TU). Each match is formed of two partners drawn from separate populations. For simplicity, we call one the "husband" and one the "wife". Husbands are drawn from a population of men indexed by $i \in \mathcal{I}$, and wives from a population of women indexed by $j \in \mathcal{J}$. We call i the *identity* of a man, as opposed to his *type* $x \in \mathcal{X}$. Identities are observed by all participants in the market. On the other hand, the econometrician only observes types, which partition the set of identities. We will write $i \in x$ or $x_i = x$ to denote that the man of identity i has type x. The corresponding notation for women will be $j \in y$ or $y_j = y$, where $y \in \mathcal{Y}$. We do not restrict the sets $\mathcal{I}, \mathcal{J}, \mathcal{X}$ and $\mathcal{Y}$ at this stage. We will denote $F_{\mathcal{I}}$ (resp. $F_{\mathcal{J}}$) the cumulative distribution function of i (resp. j.)

A match between a man i and a woman j generates a *joint surplus* $\tilde{\Phi}_{ij}$. This is shared between the two partners so that they achieve individual match surpluses $\tilde{U}_{ij}, \tilde{V}_{ij} = \tilde{\Phi}_{ij} - \tilde{U}_{ij}$, over and above the utilities they get by remaining single. We denote singlehood as "partner 0", and we will use the notation $\mathcal{X}_0 = \mathcal{X} \cup \{0\}$ and $\mathcal{Y}_0 = \mathcal{Y} \cup \{0\}$.

A *matching* is a collection of numbers $0 \leq \mu_{ij} \leq 1$ that represent the probability that man i and woman j are matched. It must be feasible:

$$\forall i, \int \mu_{ij} dF_{\mathcal{J}}(j) \leq 1 \text{ and } \forall j, \int \mu_{ij} dF_{\mathcal{I}}(i) \leq 1.$$

Our equilibrium concept is stable matching. A stable matching is a feasible matching that maximizes total joint surplus

$$\int \int \mu_{ij} dF_{\mathcal{I}}(i) dF_{\mathcal{J}}(j).$$

Associated to the feasibility constraints are multipliers $\tilde{u}_i$ and $\tilde{v}_j$; the corresponding first-order conditions are

$$\tilde{\Phi}_{ij} \leq \tilde{u}_i + \tilde{v}_j \text{ for all } i, j,$$

with equality for any match that has non-zero probability in equilibrium ($\mu_{ij} > 0$). Such an equilibrium match must split the surplus in such a way that

$$\tilde{\Phi}_{ij} = \tilde{u}_i + \tilde{v}_j$$

for some admissible values of the multipliers.

Since econometricians only observes types, their data can only consist of

- the numbers n_x (resp. m_y) of available men (resp. women) for every type $x \in \mathcal{X}$ (resp. $y \in \mathcal{Y}$)
- for each pair (x, y) the numbers μ_{xy} of matches between men of type x and women of type y.

They could also observe some statistic on the joint surplus of (x, y) matches. This could be the distribution of the joint surplus of all observed (x, y); or, less ambitiously, its average value, or some statistic like the number of divorces that brings some information on outcomes. We shall assume away such information in this paper. We will denote μ_{x0} the number of single men of type x, which is obtained by subtracting the total number of their matches from n_x.

Moreover, we will assume that *conditional on observed types*, interactions between the unobserved characteristics of the partners do not create (or destroy) joint surplus. More precisely, we state:

Assumption 1 (Separability)**.** *Equivalently:*

(i) the joint surplus from a match between man $i \in x$ and woman $j \in y$ can be written as

$$\tilde{\Phi}_{ij} = \Phi_{xy} + \varepsilon^i_y + \eta^j_x.$$

(ii) if men i and i' both have type x and women j and j' both have type y, then

$$\tilde{\Phi}_{ij} + \tilde{\Phi}_{i'j'} = \tilde{\Phi}_{ij'} + \tilde{\Phi}_{i'j}.$$

Separability was defined in Chiappori et al. (2017) and underlies the general analysis of TU models of Galichon and Salanié (2016). We refer the reader to these papers for a discussion. What matters most here is that separability allows us to decompose the two-sided equilibrum into a family of one-sided discrete choice problems with endogenous, type-dependent prices. More formally, there exists a decomposition $\Phi_{xy} = U_{xy} + V_{xy}$ of the types-driven part of joint surplus such that in equilibrium[2],

- man i is matched with a partner of the type y that maximizes $(U_{x_i t} + \varepsilon^i_t)$ over t
- woman j is matched with a partner of the type x that maximizes $(V_{z y_j} + \eta^j_z)$ over z.

The analogy with the one-sided problem is clear; but the crucial difference is that the type-dependent parameters U_{xy} and V_{xy} are endogenous, unlike the "mean utilities" of the random utility models. This will require adapting the definition of IIA.

[2]With obvious adaptations to account for unmatched partners.

2 IIA, ITS and ITA

In matching with transferable utilities, as in any equilibrium model, prices and allocations reflect value and scarcity. Intuition suggests that if for instance a man i belongs to a type x that is highly valued, in the sense that the values of Φ_{xy} are high for all potential types of partners y, then this man will do relatively well on the marriage market. Since the market-clearing prices are reflected in the decomposition $\boldsymbol{\Phi} = \boldsymbol{U} + \boldsymbol{V}$, this is simply saying that such men tend to marry women who are also highly valued, and to appropriate a large share of the joint surplus in their marriage. The same conclusion obtains if type x is relatively rare, that is if n_x is relatively small.

Now consider the ratio μ_{xy}/μ_{xt} for two types of women $y \neq t$. Dividing through by n_x, this can also be written as the ratio of the probabilities that any given man of type x will marry a woman of type y or t. We will denote $\mu^{\mathcal{X}}_{y|x}$ the probability that a man of type x marries a woman of type y (and $\mu^{\mathcal{Y}}_{x|y}$ the probability that a woman of type y marries a man of type x). Then μ_{xy}/μ_{xt} is the odds ratio for men of type x,

$$R^{\mathcal{X}}(y,t;x) = \frac{\mu^{\mathcal{X}}_{y|x}}{\mu^{\mathcal{X}}_{t|x}}.$$

In a one-sided model, IIA requires that this ratio be independent of the set of women in the marriage market, as long as this set includes some women of type y and some women of type t. The previous paragraph suggests that this cannot hold in a matching market: at a minimum, the odds ratio must depend on the numbers of women of types y and t.

Remember that the primitives of a separable matching model are the numbers of men and women of each type (n_x) and (m_y); the values of the type-dependent joint surplus (Φ_{xy}); and the distributions of the unobserved terms (ε^i_y) and (η^j_x). We propose a weaker definition of IIA:

Property 1 (IIA for separable matching with transferable utilities)**.** *Fix the parameters* (Φ_{xy}) *and the distributions of the unobserved terms* ε^i_y *and* η^j_x*.*

The model satisfies IIA if and only if for all types of men x *and* z *in* $\mathcal{X}$ *and all types of* *women y and t in $\mathcal{Y}$, the double odds ratio*

$$\frac{R^{\mathcal{X}}(y,t;x)}{R^{\mathcal{X}}(y,t;z)}$$

is independent of all subpopulation sizes (n_x) and (m_y).

Note that this ratio is simply

$$\frac{\mu_{xy}\mu_{zt}}{\mu_{xt}\mu_{zy}}.$$

Let us define the odds ratio $R^{\mathcal{Y}}(x,z;y)$ for women of type y as $\mu^{\mathcal{Y}}_{x|y}/\mu^{\mathcal{Y}}_{z|y}$. Then the double odds ratio in Property 1 can also be written as

$$\frac{R^{\mathcal{Y}}(x,z;y)}{R^{\mathcal{Y}}(x,z;t)}$$

so that the definition applies to both sides of the market.

It is not obvious a priori that there exist separable matching models in which Property 1 holds; but as we will see in section 3, the Choo and Siow (2006) model satisfies it.

In his review of Luce (1959), Debreu (1960) showed how IIA leads to counterintuitive predictions. His example used classical music recordings; we will give it in the form popularized by McFadden (1974, p 113) as the "blue bus/red bus example". In this story, commuters initially can only go to work with their car or with a blue bus; and a third of them choose to take the bus. Suppose that the bus company adds red buses to its fleet, and the population has no color preferences; then one would not expect the proportion of bus trips to change. But according to IIA, car trips should still be twice more frequent than trips with blue buses, and also than trips with red buses. This is only possible if the proportion of car trips becomes one half. To put it differently, IIA suggests that 25% of car commuters should start taking the bus simply because of a color change that (we assumed) means nothing to them.

IIA implies more specific restrictions; for instance, the elasticity of $\Pr_S(x)$ with respect to the mean utility $u(y)$ of an element y of S is the same for all $y \in S - \{x\}$. This is a very unappealing restriction to impose on a demand system. Partly as a consequence, the empirical literature moved away from the multinomial logit model; it adopted variants in which IIA does not hold, such as mixed multinomial logit. We will focus here on the kind of paradoxical conclusions implied by the blue bus-red bus example. What is really at stake in this story is that simply *splitting* options should not change market shares. What matters to commuters is the essence of a bus, not its payoff-irrelevant attributes.

In order to adapt this concept to separable matching models, we need to define precisely what a payoff-irrelevant split is. Imagine splitting woman type y into two subtypes y^1 and y^2. Payoff irrelevance of this split means three things: for all types x of men,

1. $\Phi_{xy^1} = \Phi_{xy^2} = \Phi_{xy}$
2. the vector (η^j_x) has the same distribution for all women j of type y, whether j's subtype is y^1 or y^2.
3. for any man i of type x, $\varepsilon^i_{y^1}$ and $\varepsilon^i_{y^2}$ are independent draws from the conditional distribution of ε^i_y conditional on $(\varepsilon^i_t)_{t \neq y}$.

The first requirement simply states that subtypes y^1 and y^2 generate the same type-dependent joint surplus as type y in a match with any given type x. The second rules out subtypes generating unobserved terms with different distributions. The last requirement is more subtle. It is easier to grasp when the different components of the vector (ε^i_t) are iid draws from a distribution F. Then it is simply imposing that the $\varepsilon^i_{y^1}$ and $\varepsilon^i_{y^2}$ terms be independent draws from that same F.

By definition, payoff-irrelevant splits are generated by sequences of such elemental splits. We can now state

Property 2 (Independence of Irrelevant Type Splits)**.** *Splitting types in payoff-irrelevant ways should not change choice probabilities.*

The ITS requirement is satisfied by a class of models which extends the Random Scalar Coefficient models introduced in Galichon–Salanié (2016). Define *Generalized Random Co-*

efficients models (GRC) as models such that there exist real-valued random variables ε_i and η_j and functions ζ_{xy} and ξ_{xy} with

$$\forall x \in \mathcal{X}, \forall y \in \mathcal{Y}_0,\ \varepsilon^i_y = \zeta_{xy}(\varepsilon_i) \text{ and } \forall y \in \mathcal{Y}, \forall x \in \mathcal{X}_0, \eta^j_x = \xi_{xy}(\eta_j),$$

In these models, the random utilities associated to any pair of alternatives are perfectly dependent. GRC models can be viewed as an extension of the random coefficient models popular in empirical IO, with the important restriction that separability rules out an "idiosyncratic" term of the form w_{ij}. The GRC models extend the Random Scalar Coefficients models of Galichon–Salanié (2016), in which the functions ζ_{xy} and ξ_{xy} are linear and the random variables ε_i and η_j are scalar.

It is easily seen that Generalized Random Coefficients models satisfy ITS, and that they do *not* satisfy IIA. We leave it as an open problem whether any separable matching model that satisfies ITS is a GRC model.

The blue bus-red bus example in the next section shows that the multinomial logit model, which is the only one to satisfy IIA in one-sided choice models, violates ITS. Matching markets are two-sided by their very nature; this is reflected in the endogeneous nature of the decomposition $\boldsymbol{\Phi} \equiv \boldsymbol{U} + \boldsymbol{V}$. Still, it is not hard to construct illustrations similar to Debreu's example. Because "mean utilities" are endogenous, notation and characterization take more work; but the intuition is similar.

If a matching model is given as $\mu_{xy} = F_{xy}(\boldsymbol{\Phi}, \boldsymbol{n}, \boldsymbol{m})$, we define the property of *Irrelevance to Type Aggregation* (ITA) as follows: for any $x_1, x_2 \in \mathcal{X}$ and $y_1, y_2 \in \mathcal{Y}$ and $\boldsymbol{\Phi}, \boldsymbol{\Phi}'$, $\boldsymbol{n}, \boldsymbol{n}'$, and $\boldsymbol{m}, \boldsymbol{m}'$ such that:

- $\Phi_{x_i y_j} = \Phi'_{x_k y_l}$ for any $i, j, k, l \in \{1, 2\}$, and
- $\Phi_{xy} = \Phi'_{xy}$ for any $x \notin \{x_1, x_2\}$ and $y \notin \{y_1, y_2\}$, and
- $n_{x_1} + n_{x_2} = n'_{x_1} + n'_{x_2}$ and $m_{y_1} + m_{y_2} = m'_{y_1} + m'_{y_2}$, and
- $n_x = n'_x$ for any $x \notin \{x_1, x_2\}$ and $m_y = m'_y$ for any $y \notin \{y_1, y_2\}$,

then, letting $\boldsymbol{\mu} = F(\boldsymbol{\Phi}, \boldsymbol{n}, \boldsymbol{m})$ and $\boldsymbol{\mu}' = F(\boldsymbol{\Phi}', \boldsymbol{n}', \boldsymbol{m}')$, one has:

- $\sum_{i\in\{1,2\}} \sum_{j\in\{1,2\}} \mu_{x_i y_j} = \sum_{i\in\{1,2\}} \sum_{j\in\{1,2\}} \mu'_{x_i y_j}$, and
- $\mu_{xy} = \mu'_{xy}$ for any $x \notin \{x_1, x_2\}$ and $y \notin \{y_1, y_2\}$.

The meaning of ITA is very intuitive: it means that we can *aggregate* types x_1 and x_2 into a single type denoted x_{12} and types y_1 and y_2 into a single type denoted y_{12}, and that

$$\mu_{x_{12}y_{12}} = \mu_{x_1y_1} + \mu_{x_1y_2} + \mu_{x_2y_1} + \mu_{x_2y_2}$$

does not depend on the relative composition of type x_{12} in types x_1 and x_2, nor does it depend on the relative composition of type y_{12} in types y_1 and y_2.

3 A modified blue-bus/red-bus example

Choo and Siow (2006) added to the separable model two assumptions: that markets are "large" and that the unobserved terms of Assumption 1 are iid draws from a standard type-I extreme value distribution.

They showed that it generates a very convenient multinomial logit form for the separable model. More precisely,

$$\mu_{y|x} = \frac{\exp(U_{xy})}{\sum_t \exp(U_{xt})}$$

where the (U_{xt}) are the equilibrium quantities defined in section 1. In addition, in all models of this class we have

$$\frac{\mu_{xy}\mu_{zt}}{\mu_{xt}\mu_{zy}} = \exp\left(\left(\Phi_{xy} + \Phi_{zt} - \Phi_{xt} - \Phi_{zy}\right)/2\right),$$

which is independent of the numbers of men and women. As a consequence, Property 1 holds in all of them.

Proposition 1. *Every model of the Choo–Siow family satisfies IIA as defined in Property 1.*

Note that the Choo and Siow model is not the only separable model that satisfies IIA. Consider a separable model where the heterogeneity is a nested logit on both sides of the market, with only two nests: one for singlehood, another one for all other marital options. The coefficient of the nests is λ on the side of men, γ on the side of women; the Choo-Siow model obtains for $\lambda = \gamma = 1$. It is not hard to see that

$$\mu_{xy} = \frac{\mu_{x0}^{1/(\lambda+\gamma)}}{\left(m_y - \mu_{0y}\right)^{\frac{1-\lambda}{\lambda+\gamma}}} \frac{\mu_{0y}^{1/(\lambda+\gamma)}}{\left(m_y - \mu_{0y}\right)^{\frac{1-\gamma}{\lambda+\gamma}}} \exp\left(\frac{\Phi_{xy}}{\lambda+\gamma}\right).$$

This implies that $\log \mu_{xy} - \frac{\Phi_{xy}}{\lambda+\gamma}$ is additively separable between x and y;

$$\frac{\mu_{xy}\mu_{zt}}{\mu_{xt}\mu_{zy}} = \exp\left(\left(\Phi_{xy} + \Phi_{zt} - \Phi_{xt} - \Phi_{zy}\right)/(\lambda+\gamma)\right),$$

and this model too satisfies IIA. We have not found any other separable model for which IIA holds.

The Choo and Siow framework has a number of strong implications. In particular, it violates ITS and ITA. We show this by way of a simple example. Let x and y consist of education, with two levels C (college) and N (no college), and suppose that the matrix $\mathbf{\Phi}$ of Assumption 1 has

$$\exp(\Phi_{NN}/2) = a\ ;\ \exp(\Phi_{NC}/2) = \exp(\Phi_{CN}/2) = b\ ;\ \exp(\Phi_{CC}/2) = c,$$

where a, b, c are arbitrary positive numbers.

Call this the *original model.* Now let us distinguish two types of college graduates: those (C_e) whose Commencement fell on an even-numbered day and those (C_o) for whom it was on an odd-numbered day. We will assume that this difference is payoff-irrelevant, so that

$$\exp(\Phi_{NC_i}/2) = \exp(\Phi_{C_in}/2) = a \ \text{ for } i = e, o$$

and $\exp(\Phi_{C_iC_j}) = c$ for $i, j = e, o$. We will also assume that the population of college graduates is split evenly across Commencement days: $n_{C_e} = n_{C_o}$ and $m_{C_e} = m_{C_o}$.

We now show that adding the irrelevant Commencement distinction in this *revised model* is "equivalent" to changing the joint surplus Φ of the original model. More precisely:

Proposition 2. *In the revised model, define* $\mu_{CC} = \sum_{i,j=e,o} \mu_{C_i C_j}$ *the total number of matches between college-educated partners;* $\mu_{CN} = \mu_{C_e,N} + \mu_{C_o,N}$ *the total number of matches between college-educated men and non-college women (and symmetrically* μ_{NC}*); and* $\mu_{C0} = \mu_{C_e,0} + \mu_{C_o,0}$ *the total number of college-educated men who remain single (and symmetrically* μ_{0C}*).*

In equilibrium, these numbers are identical to the equilibrium matching patterns of the original model after substituting $\mathbf{\Phi}'$ *to* $\mathbf{\Phi}$*, where*

$$\exp(\Phi'_{NN}/2) = a \; ; \; \exp(\Phi'_{NC}/2) = \exp(\Phi'_{CC}/2) = b\sqrt{2} \; ; \; \exp(\Phi'_{CN}/2) = 2c.$$

Proof of Proposition 2. The results in Choo and Siow show that given numbers (n_C, n_N, m_C, m_N) of men and women, the equilibrium in the original model has

$$\begin{aligned}
n_C &= \mu_{C0} + \mu_{CN} + \mu_{CC} && (3.1)\\
n_N &= \mu_{N0} + \mu_{NN} + \mu_{NC}\\
m_C &= \mu_{0C} + \mu_{NC} + \mu_{CC}\\
m_N &= \mu_{0N} + \mu_{NN} + \mu_{CN}\\
\mu_{CN} &= b\sqrt{\mu_{C0}\mu_{0N}} && (3.2)\\
\mu_{NC} &= b\sqrt{\mu_{N0}\mu_{0C}}\\
\mu_{CC} &= c\sqrt{\mu_{C0}\mu_{0C}} && (3.3)\\
\mu_{NN} &= a\sqrt{\mu_{N0}\mu_{0N}}.
\end{aligned}$$

Take college-educated men for instance. Equation (3.1) requires that the number of college-educated men who remain single and who marry any type of woman must add up to the number of college-educated men available. Substituting equations (3.2) and (3.3) gives

$$\mu_{C0} + \left(b\sqrt{\mu_{0N}} + c\sqrt{\mu_{0C}}\right)\sqrt{\mu_{C0}} = n_C.$$

As noted by Graham (2013) and Decker et al. (2012), this is a quadratic equation in $\sqrt{\mu_{C0}}$, given the numbers of single women μ_{0N} and μ_{0C}. They showed that the whole system can be rewritten as the four coupled quadratic equations in the square roots of the numbers of singles:

$$\begin{aligned}
n_C &= \mu_{C0} + \left(b\sqrt{\mu_{0N}} + c\sqrt{\mu_{0C}}\right)\sqrt{\mu_{C0}} \\
n_N &= \mu_{N0} + \left(a\sqrt{\mu_{0N}} + b\sqrt{\mu_{0C}}\right)\sqrt{\mu_{N0}} \\
m_C &= \mu_{0C} + \left(b\sqrt{\mu_{N0}} + c\sqrt{\mu_{C0}}\right)\sqrt{\mu_{0C}} \\
m_N &= \mu_{0N} + \left(a\sqrt{\mu_{N0}} + b\sqrt{\mu_{C0}}\right)\sqrt{\mu_{0N}}.
\end{aligned} \tag{3.4}$$

Now let us introduce the payoff-irrelevant Commencement distinction. Under our assumptions, the quadratic equation that defines the equilibrium for college-educated men whose Commencement was on an even day take the form

$$\mu_{C_e0} + \sqrt{\mu_{C_e0}}\left(c\left(\sqrt{\mu_{0C_e}} + \sqrt{\mu_{0C_o}}\right) + b\sqrt{\mu_{0N}}\right) = n_{C_e} = n_C/2.$$

and that for college-educated men whose Commencement was on an odd day is

$$\mu_{C_o0} + \sqrt{\mu_{C_o0}}\left(c\left(\sqrt{\mu_{0C_e}} + \sqrt{\mu_{0C_o}}\right) + b\sqrt{\mu_{0N}}\right) = n_{C_o} = n_C/2.$$

These two equations are identical; and since they have only one feasible root, they imply $\mu_{C_e0} = \mu_{C_o0}$. Similarly, $\mu_{0C_e} = \mu_{0C_o}$. For men as for women, there are just as many single college graduates in both Commencement groups.

This in turn implies that $\mu_{C_iC_j} = c\sqrt{\mu_{C_i0}\mu_{0C_j}}$ cannot depend on $i, j = e, o$; and that $\mu_{C_iN} = b\sqrt{\mu_{C_i0}\mu_{0N}}$ and μ_{NC_i} cannot depend on $i = e, o$.

Putting things together, and using the notation defined in the statement of the Proposition, we obtain

$$\begin{aligned}
\mu_{CC} &= 4c\sqrt{\frac{\mu_{C0}}{2}\frac{\mu_{0C}}{2}} = 2c\sqrt{\mu_{C0}\mu_{0C}} \\
\mu_{CN} &= 2b\sqrt{\frac{\mu_{C0}}{2}\mu_{0N}} = b\sqrt{2}\sqrt{\mu_{C0}\mu_{0N}} \\
\mu_{NC} &= 2b\sqrt{\mu_{N0}\frac{\mu_{0C}}{2}} = b\sqrt{2}\sqrt{\mu_{N0}\mu_{0C}} \\
\mu_{NN} &= a\sqrt{\mu_{N0}\mu_{0N}}.
\end{aligned}$$

This gives the equilibrium equations

$$\begin{aligned} n_C &= \mu_{C0} + b\sqrt{2}\sqrt{\mu_{C0}\mu_{0N}} + 2c\sqrt{\mu_{C0}\mu_{0C}} \\ n_N &= \mu_{N0} + a\sqrt{\mu_{N0}\mu_{0N}} + b\sqrt{2}\sqrt{\mu_{N0}\mu_{0C}} \end{aligned} \tag{3.5}$$

and two symmetric equations for women. But the system (3.4) had

$$\begin{aligned} n_C &= \mu_{C0} + b\sqrt{\mu_{C0}\mu_{0N}} + c\sqrt{\mu_{C0}\mu_{0C}} \\ n_N &= \mu_{N0} + a\sqrt{\mu_{N0}\mu_{0N}} + b\sqrt{\mu_{N0}\mu_{0C}}. \end{aligned} \tag{3.6}$$

Comparing (3.5) and (3.6) shows that adding the irrelevant Commencement distinction changes the equilibrium matching patterns as if we had changed the joint surplus $\mathbf{\Phi}$ to $\mathbf{\Phi}'$. ∎

As argued by Siow (2015) and Chiappori et al. (2017), a natural measure of complementarity in this model is the "supermodularity module" of the $\mathbf{\Phi}$ matrix,

$$\mathcal{D} = \Phi_{CC} + \Phi_{NN} - \Phi_{CN} - \Phi_{NC}.$$

Its monotonic transform $\mathcal{C} = \exp(\mathcal{D}/2)$ is independent of the subpoulation sizes, and it equals the double odds ratio of matching patterns in equilibrium defined in property 1:

$$\mathcal{C} = \frac{\mu_{CC}\mu_{NN}}{\mu_{CN}\mu_{NC}},$$

Therefore this model satisfies IIA as we defined it. In fact all models of the Choo and Siow family have

$$\frac{\mu_{xy}\mu_{zt}}{\mu_{xt}\mu_{zy}} = \exp\left(\left(\Phi_{xy} + \Phi_{zt} - \Phi_{xt} - \Phi_{zy}\right)/2\right),$$

so that Property 1 holds in all of them.

Going back to our example, it is easy to check that substituting $\mathbf{\Phi}'$ to $\mathbf{\Phi}$ does not affect $\mathcal{C} = ac/b^2$. On the other hand, it does change equilibrium marriage patterns. Suppose for instance that $n_C = m_C$ and $n_N = m_N$: there are as many men as women in either education group. Then all equations are symmetric in gender, and we must have $\mu_{C0} = \mu_{0C}$

and $\mu_{N0} = \mu_{0N}$, both in the original and in the revised model. The equations for men in the original model simplify to:

$$\begin{aligned} n_C &= \mu_{C0}(1+c) + b\sqrt{\mu_{N0}\mu_{C0}} \\ n_N &= \mu_{N0}(1+a) + b\sqrt{\mu_{N0}\mu_{C0}}. \end{aligned}$$

Suppose moreover that the college-educated are half of the population in each gender: $n_N = n_C \equiv n$ and $m_N = m_C \equiv m$, so that by subtraction $\mu_{C0}(1+c) = \mu_{N0}(1+a)$. Then we obtain in the original model

$$\begin{aligned} \mu_{C0} = \mu_{0C} &= \sqrt{\frac{1+a}{1+c}} \frac{n}{b + \sqrt{(1+a)(1+c)}} \\ \mu_{N0} = \mu_{0N} &= \sqrt{\frac{1+c}{1+a}} \frac{n}{b + \sqrt{(1+a)(1+c)}} \\ \mu_{NN} &= a\mu_{N0} \\ \mu_{NC} = \mu_{CN} &= \frac{bn}{b + \sqrt{(1+a)(1+c)}} \\ \mu_{CC} &= c\mu_{C0}. \end{aligned}$$

In the revised model, $b \to b\sqrt{2}$ and $c \to 2c$. Remember that a, b and c are exponentials and therefore positive. Since μ_{C0} is a decreasing function of both b and c, it must be lower in the revised model. μ_{NC} is an increasing function of $b/\sqrt{1+c}$; but

$$b\sqrt{2}/\sqrt{1+2c} > b/\sqrt{1+c}$$

and μ_{NC} must be higher. Since $\mu_{N0} + \mu_{NN} + \mu_{NC}$ is fixed at m and $\mu_{NN}/\mu_{N0} = a$ is unchanged, it follows that both μ_{NN} and μ_{N0} must be lower. Finally,

$$\mu_{CC} = n - \mu_{C0} - \mu_{CN} = \frac{n}{b + \sqrt{(1+a)(1+c)}} \left(\sqrt{1+c} - \frac{1}{\sqrt{1+c}} \right) = \frac{n}{b + \sqrt{(1+a)(1+c)}} \frac{c}{\sqrt{1+c}}.$$

We have just seen that $\frac{c}{\sqrt{1+c}}$ increases by a factor of more than $\sqrt{2}$. The denominator $b+\sqrt{(1+a)(1+c)}$ increases by less, since $\sqrt{1+c}$ increases by a factor smaller than $\sqrt{2}$ and nb by just $\sqrt{2}$. Therefore μ_{CC} must increase.

To recapitulate:

- There are fewer college graduate singles. This is the equivalent of more people taking the bus in Debreu 1960: mere payoff-irrelevant splits increase probabilities of choice.
- More suprisingly, there are also fewer non-college singles; but the fall in singles is smaller than for college graduates.
- There are more matches between N and C, fewer between N and N, and more between C and C.
- Since the expected utility is simply minus the logarithm of the probability of singlehood in the CS model, expected utilities increase at each level of education.

These are clearly unappealing properties: since the Commencement date is irrelevant to all market participants, a "more reasonable" model would imply none of these changes. This example shows that the Choo and Siow model violates Irrelevance of Type Splits, as splitting college graduation according to the payoff-irrelevant parity of the Commencement day changes matching patterns μ and expected utilities for college graduates. By the same token, one sees that this model also violates Irrelevance of Type Aggregation.

4 Remarks

4.1 A stronger notion of IIA

A valid criticism of the notion of IIA we have adopted in property 1 is that since it does not include the option of remaining unmatched, it leaves out one of the marital options. To remedy this, we could introduce the strong-IIA property as follows:

Property 3. *Fix the parameters* (Φ_{xy}) *and the distributions of the unobserved terms* ε_y^i *and* η_x^j.

The model satisfies strong-IIA *if and only if the following two sets of conditions are met:*

(i) for all types of men x *and* z *in* $\mathcal{X}$ *and all men's marital options* y *and* t *in* $\mathcal{Y}_0$, *the double odds ratio*

$$\frac{R^{\mathcal{X}}(y,t;x)}{R^{\mathcal{X}}(y,t;z)} = \frac{\mu_{xy}\mu_{zt}}{\mu_{xt}\mu_{zy}}$$

is independent of all subpopulation sizes (n_x) *and* (m_y)*; and*

(ii) for all types of women y *and* t *in* $\mathcal{Y}$ *and all women's marital options* x *and* z *in* $\mathcal{X}_0$, *the double odds ratio*

$$\frac{R^{\mathcal{Y}}(x,z;y)}{R^{\mathcal{Y}}(x,z;t)} = \frac{\mu_{xy}\mu_{zt}}{\mu_{xt}\mu_{zy}}$$

is independent of all subpopulation sizes (n_x) *and* (m_y).

It is now straighforward to see that Choo and Siow's model does *not* satisfy strong-IIA. Indeed, in Choo and Siow's model, take $y = 0$; one has

$$\frac{\mu_{xy}\mu_{zt}}{\mu_{xt}\mu_{zy}} = \frac{\mu_{x0}}{\mu_{z0}}\frac{\mu_{zt}}{\mu_{xt}} = \sqrt{\frac{\mu_{x0}}{\mu_{z0}}}\exp\left(\Phi_{zt} - \Phi_{zt}\right)$$

which clearly depends on the subpopulations sizes. Neither do Generalized Random Coefficients models satisfy strong-IIA, as it is stronger than IIA.

4.2 Dagsvik's non-separable model

Dagsvik (2000) proposes a model which is not separable, and does not have constant returns to scale. This model has

$$\mu_{xy} = \mu_{x0}\mu_{0y}\exp(\Phi_{xy}),$$

where μ_{x0} and μ_{0y} are adjusted by the marginal constraints

$$\begin{aligned} n_x &= \mu_{x0} + \sum_{y \in \mathcal{Y}} \mu_{x0}\mu_{0y} \exp(\Phi_{xy}) \\ m_y &= \mu_{0y} + \sum_{x \in \mathcal{X}} \mu_{x0}\mu_{0y} \exp(\Phi_{xy}) \end{aligned}$$

In contrast, in Choo and Siow, the product $\mu_{x0}\mu_{0y}$ appears with a square root in the expression of μ_{xy}, which yields constant returns to scale.

The following proposition is easy to verify:

Proposition 3. *Dagsvik's model verifies both strong-IIA and ITA.*

Whether Dagsvik is the only matching model that verifies both strong-IIA and ITA is left as an open question.

References

Arrow, K. (1959): "Rational Choice Functions and Orderings," *Economica*, 26, 121–127.

Chiappori, P.-A., Salanié, B., and Y. Weiss (2017), "Partner Choice, Investment in Children, and the Marital College Premium", *American Economic Review*, 107, 2109–2167.

Choo, E. and A. Siow (2006). "Who Marries Whom and Why", *Journal of Political Economy*, 114, 175–201.

Dagsvik, J. (2000). "Aggregation in matching markets", *International Economic Review*, 41, 27–58.

Decker, C., Lieb, E., McCann, R., and B. Stephens (2012): "Unique Equilibria and Substitution Effects in a Stochastic Model of the Marriage Market," *Journal of*

Economic Theory, 148, 778–792.

Fudenberg, D., Ijima, R., and T. Strzalecki (2015): "Stochastic Choice and Revealed Perturbed Utility", *Econometrica*, 83, 2371–2409.

Galichon, A. and B. Salanié (2016), "Cupid's Invisible Hand: Social Surplus and Identification in Matching Models", mimeo Columbia.

Graham, B. (2013), "Uniqueness, comparative static, and computational methods for an empirical one-to-one transferable utility matching model," *Structural Econometric Models*, 31, 153–181.

Luce, D. (1959): *Individual Choice Behavior.* New York: Wiley.

Luce, D. and H. Raiffa (1957): *Games and Decisions.* New York: Wiley.

McFadden, D. (1974), "Conditional Logit Analysis of Qualitative Choice Behaviour", in *Frontiers of Econometrics*, P. Zarembka ed, Academic Press: New York.

Nash, J. (1950): "The Bargaining Problem", *Econometrica*, 18, 155–162.

Siow, A. (2015),"Testing Becker's Theory of Positive Assortative Matching", *Journal of Labor Economics*, 33, 409–441.